MARTINGALE DIFFERENCES AND THE METRIC THEORY OF CONTINUED FRACTIONS

ALAN K. HAYNES AND JEFFREY D. VAALER

Abstract. We investigate a collection of orthonormal functions that encodes information about the continued fraction expansion of real numbers. When suitably ordered these functions form a complete system of martingale differences and are a special case of a class of martingale differences considered by R. F. Gundy. By applying known results for martingales we obtain corresponding metric theorems for the continued fraction expansion of almost all real numbers.

1. Introduction

Throughout this paper we work with real valued functions defined on the compact group $\mathbb{R}/\mathbb{Z}$. As usual we regard such functions as defined on $\mathbb{R}$ and having period 1. We frequently regard $\mathbb{R}/\mathbb{Z}$ as a probability space with respect to a normalized Haar measure defined on the σ-algebra of Borel subsets, or restricted to a finite sub-σ-algebra. We also work with elements of the torsion subgroup $\mathbb{Q}/\mathbb{Z}$. If β is a point in $\mathbb{Q}/\mathbb{Z}$ we write $\beta = a/q$ where q is a positive integer and a is an integer representing a unique reduced residue class modulo q. By the *height* of β we understand the positive integer $h(\beta) = q$, which is also the order of β in $\mathbb{Q}/\mathbb{Z}$. In Section 2 we define a countable collection of functions $f_\beta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$, indexed by points β in $\mathbb{Q}/\mathbb{Z}$. These functions form a complete orthonormal basis for the Hilbert space $L^2(\mathbb{R}/\mathbb{Z})$ and also encode information about continued fractions. The functions f_β, which are the subject of this paper, were used by Hata [9] in a slightly different form to obtain interesting identities for sums over Farey fractions. In particular, our Theorem 4 is similar to [9, Lemma 3.1]. We will show that for certain natural orderings the functions f_β, together with a corresponding sequence of finite σ-algebras, form a sequence of martingale differences. And we will show that the value of $f_\beta(\alpha)$ is determined in an elementary way by the convergents and intermediate convergents from the continued fraction expansion of α.

2000 *Mathematics Subject Classification.* 11B57, 11K50, 60G46.

Key words and phrases. Farey fractions, continued fractions, martingales.

Recall that each irrational real number α has an infinite simple continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [a_0; a_1, a_2, a_3, \dots],$$

where a_0 is an integer and $a_1, a_2, \dots$ is a sequence of positive integers uniquely determined by α. Here we adopt standard notation and terminology as developed in [10], [11], or [12]. The number a_n is the nth partial quotient of α. If α is rational we write

$$\alpha = [a_0; a_1, a_2, \dots, a_N]$$

for one of its two finite continued fraction expansions. The principal convergents from the continued fraction expansion of an irrational real number α are defined by setting $p_{-2} = 0$, $q_{-2} = 1$, $p_{-1} = 1$, $q_{-1} = 0$, and then by the recursive formulas

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2} \tag{1}$$

for $n = 0, 1, 2, \dots$. Of course $a_n = a_n(\alpha)$, $p_n = p_n(\alpha)$, and $q_n = q_n(\alpha)$ depend on α, but to simplify notation we often suppress this dependence.

If α is an irrational point in $\mathbb{R}/\mathbb{Z}$, that is, α does not belong to $\mathbb{Q}/\mathbb{Z}$, then the partial quotient a_0 is not uniquely determined. For our purposes it will be convenient to set $a_0 = 0$ and so to identify α in $\mathbb{R}/\mathbb{Z}$ with its coset representative in the open interval $(0, 1)$. Then we also have $p_0 = 0$ and $q_0 = 1$. We will make use of the convergents and the intermediate convergents from the continued fraction expansion of α. It will be convenient to organize these by defining

$$E_n = \left\{ \frac{mp_{n-1} + p_{n-2}}{mq_{n-1} + q_{n-2}} : m = 1, 2, \dots, a_n \right\} \tag{2}$$

for $n = 1, 2, \dots$. Each set E_n contains a_n distinct fractions, including the principal convergent p_n/q_n. The remaining fractions (if any) indexed by $m = 1, 2, \dots, a_n - 1$ are the intermediate convergents to α. It is easy to check that

$$E_n = \big\{ [0; a_1, a_2, \dots, a_{n-1}, m] : m = 1, 2, \dots, a_n \big\}. \tag{3}$$

Again the set $E_n = E_n(\alpha)$ depends on α, but we often suppress this dependence.

For each point β in $\mathbb{Q}/\mathbb{Z}$ the function $f_\beta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is a step function taking at most three distinct values on intervals of positive measure, and defined below by (13) or (15). If α is an irrational point in $\mathbb{R}/\mathbb{Z}$ then the value of $f_\beta(\alpha)$ is also determined by the convergents and intermediate convergents to α. The precise result is as follows.

Theorem 1. *Let α be an irrational point in $\mathbb{R}/\mathbb{Z}$, and for $n = 1, 2, \ldots$ let E_n be the collection of convergents and intermediate convergents defined by* (2)*. If $\beta \in \mathbb{Q}/\mathbb{Z}$ then $f_\beta(\alpha) \neq 0$ if and only if β belongs to E_n for some $n = 1, 2, \ldots$. Moreover, if β belongs to E_n then*

$$f_\beta(\alpha) = (-1)^{n-1} q_{n-1}(\alpha). \tag{4}$$

Further identities involving partial sums of the functions f_β are given in Section 3.

By combining Theorem 1 with a convergence theorem for martingale differences due to R. F. Gundy [6, Theorem 2.1(a)], we obtain the following metric theorem.

Theorem 2. *Let $F : \mathbb{R}/\mathbb{Z} \to \mathbb{R} \cup \{\pm\infty\}$ be a Borel measurable function that is finite almost everywhere. Then there exist real numbers $\{c(\beta) : \beta \in \mathbb{Q}/\mathbb{Z}\}$ such that*

$$\lim_{N \to \infty} \sum_{n=1}^{N} (-1)^{n-1} q_{n-1}(\alpha) \sum_{\beta \in E_n(\alpha)} c(\beta) = F(\alpha) \tag{5}$$

for almost all irrational points α in $\mathbb{R}/\mathbb{Z}$.

The numbers $\{c(\beta) : \beta \in \mathbb{Q}/\mathbb{Z}\}$ that occur in the statement of Theorem 2 are not uniquely determined by the measurable function F. In particular, there exist $c(\beta)$ that are not all zero but for which the limit (5) is zero for almost all α. We give an example of this in Section 4. An interesting feature of Theorem 2 is that no assumption is made concerning the integrability of F. If we assume that the function F is in $L^1(\mathbb{R}/\mathbb{Z})$ then the conclusion (5) is much easier to prove. In fact, if F is in $L^1(\mathbb{R}/\mathbb{Z})$ then there is a unique choice of the numbers $\{c(\beta) : \beta \in \mathbb{Q}/\mathbb{Z}\}$ such that (5) converges almost everywhere *and* in L^1-norm to the function F. If F is in $L^2(\mathbb{R}/\mathbb{Z})$ then there is a unique choice of the numbers $\{c(\beta) : \beta \in \mathbb{Q}/\mathbb{Z}\}$ such that (5) converges almost everywhere *and*

$$\sum_{\beta \in \mathbb{Q}/\mathbb{Z}} c(\beta)^2 < \infty.$$

This follows from the fact (see Theorem 4) that the collection of functions $\{f_\beta : \beta \in \mathbb{Q}/\mathbb{Z}\}$ forms a complete orthonormal basis for $L^2(\mathbb{R}/\mathbb{Z})$.

In Section 5 we assume that numbers $\{c(\beta) : \beta \in \mathbb{Q}/\mathbb{Z}\}$ are given and we consider the behavior of the corresponding partial sums, such as occur on the left of (5). If the partial sums are bounded in L^1-norm then it is an immediate consequence of the martingale convergence theorem that the partial sums converge almost everywhere. We report this as Theorem 9. If we assume that the map $\beta \mapsto c(\beta)h(\beta)$ is bounded on $\mathbb{Q}/\mathbb{Z}$, then we can draw further conclusions about the set of irrational points α where the partial sums converge.

Theorem 3. *Let $\{c(\beta) : \beta \in \mathbb{Q}/\mathbb{Z}\}$ be a collection of real numbers such that $\beta \mapsto c(\beta)h(\beta)$ is bounded on $\mathbb{Q}/\mathbb{Z}$. For each irrational point α in $\mathbb{R}/\mathbb{Z}$ and positive integer Q let $M = M(\alpha, Q)$ and $N = N(\alpha, Q)$ be the unique positive integers such that*

$$1 \le M \le a_N \quad \text{and} \quad Mq_{N-1} + q_{N-2} \le Q < (M+1)q_{N-1} + q_{N-2}.$$

Write $\mathcal{C}$ for the subset of irrational points α in $\mathbb{R}/\mathbb{Z}$ such that

$$\lim_{Q\to\infty} \sum_{n=1}^{N} (-1)^{n-1} q_{n-1}(\alpha) \sum_{\substack{\beta \in E_n(\alpha) \\ h(\beta) \le Q}} c(\beta) \tag{6}$$

exists and is finite. Write $\mathcal{D}$ for the subset of irrational points α in $\mathbb{R}/\mathbb{Z}$ such that both

$$\liminf_{Q\to\infty} \sum_{n=1}^{N} (-1)^{n-1} q_{n-1}(\alpha) \sum_{\substack{\beta \in E_n(\alpha) \\ h(\beta) \le Q}} c(\beta) = -\infty \tag{7}$$

and

$$\limsup_{Q\to\infty} \sum_{n=1}^{N} (-1)^{n-1} q_{n-1}(\alpha) \sum_{\substack{\beta \in E_n(\alpha) \\ h(\beta) \le Q}} c(\beta) = +\infty. \tag{8}$$

Write $\mathcal{E}$ for the subset of irrational points α in $\mathbb{R}/\mathbb{Z}$ such that

$$\sum_{n=1}^{\infty} q_{n-1}(\alpha)^2 \sum_{\beta \in E_n(\alpha)} c(\beta)^2 < \infty.$$

Then we have

$$\text{(i) } |\mathcal{C} \cup \mathcal{D}| = 1, \quad \text{(ii) } |\mathcal{C} \setminus \mathcal{E}| = 0, \quad \textit{and} \quad \text{(iii) } |\mathcal{E} \setminus \mathcal{C}| = 0. \tag{9}$$

We note that the restriction $h(\beta) \le Q$ in (6), (7), and (8), effects only the term for which $n = N$. Obviously $N \to \infty$ as $Q \to \infty$, but in a manner that depends on α.

We would like to thank H. L. Montgomery for first calling our attention to the functions f_β.

2. A Complete System of Orthonormal Functions

We will work with functions $g : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ of bounded variation that satisfy the condition

$$g(x) = \tfrac{1}{2}g(x+) + \tfrac{1}{2}g(x-) \tag{10}$$

at each point x. When (10) is satisfied we say that the function g is *normalized*. The collection of all normalized real (or complex) valued functions of bounded

variation on $\mathbb{R}/\mathbb{Z}$ is a real (or complex) vector space. An important example is the sawtooth function $\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ defined by

$$\psi(x) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not in } \mathbb{Z}, \\ 0 & \text{if } x \text{ is in } \mathbb{Z}, \end{cases}$$

where $[x]$ is the integer part of x.

For each positive integer Q we define $\mathcal{F}_Q$ to be the finite set

$$\mathcal{F}_Q = \{\beta \in \mathbb{Q}/\mathbb{Z} : h(\beta) \leq Q\}.$$

It follows that $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q$ is the union of exactly

$$\bigl|\mathcal{F}_Q\bigr| = \sum_{q \leq Q} \varphi(q)$$

component intervals, where φ is the Euler φ-function. Clearly each component interval determines a unique left hand endpoint β_1 in $\mathcal{F}_Q$ and a unique right hand endpoint β_2 in $\mathcal{F}_Q$. In this case it will be convenient to write $I(\beta_1, \beta_2)$ for the corresponding (open) component interval and $\overline{I}(\beta_1, \beta_2)$ for its closure in $\mathbb{R}/\mathbb{Z}$. We say that the elements of the ordered set $\{\beta_1, \beta_2\}$ are *consecutive* points in $\mathcal{F}_Q$ if there exists a component interval of the form $I(\beta_1, \beta_2)$ in $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q$. More generally, if β_1 and β_2 are points in $\mathbb{Q}/\mathbb{Z}$ we write $I(\beta_1, \beta_2)$ for the corresponding component interval whenever the elements of the ordered set $\{\beta_1, \beta_2\}$ are consecutive points in $\mathcal{F}_Q$ for some positive integer Q. We note that the *normalized* characteristic function of the component interval $I(\beta_1, \beta_2)$ is given by

$$\bigl|I(\beta_1, \beta_2)\bigr| + \psi(\beta_1 - x) + \psi(x - \beta_2),$$

where

$$\bigl|I(\beta_1, \beta_2)\bigr| = h(\beta_1)^{-1} h(\beta_2)^{-1}$$

is the Haar measure of $I(\beta_1, \beta_2)$.

Now suppose that β is a nonzero point in $\mathbb{Q}/\mathbb{Z}$ such that $h(\beta) = Q$. Then there exists a unique point β' in $\mathcal{F}_Q$ such that $\{\beta', \beta\}$ are consecutive points in $\mathcal{F}_Q$, and there exists a unique point β'' in $\mathcal{F}_Q$ such that $\{\beta, \beta''\}$ are consecutive points in $\mathcal{F}_Q$. Thus we have two well defined maps $\beta \mapsto \beta'$ and $\beta \mapsto \beta''$ from $\mathbb{Q}/\mathbb{Z} \setminus \{0\}$ into $\mathbb{Q}/\mathbb{Z}$. It is easy to verify that these maps are both surjective. And they satisfy the basic identities

$$h(\beta) = h(\beta') + h(\beta''), \quad \gcd\{h(\beta'), h(\beta)\} = 1, \quad \gcd\{h(\beta), h(\beta'')\} = 1, \tag{11}$$

and also

$$h(\beta - \beta') = h(\beta)h(\beta') \quad \text{and} \quad h(\beta'' - \beta) = h(\beta)h(\beta''). \tag{12}$$

We will often use the fact that if β is a nonzero point in $\mathbb{Q}/\mathbb{Z}$ then $I(\beta', \beta'')$ is a component interval of $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_q$ for all q such that $\max\{h(\beta'), h(\beta'')\} \leq q < h(\beta)$. Of course the remarks and notation we have developed here reflect well known properties of Farey fractions (see [7] or [10]), but modified slightly to

account for our working in the group $\mathbb{Q}/\mathbb{Z}$. The following result also follows easily from basic properties of Farey fractions.

Lemma 1. *Suppose that β and γ are distinct nonzero points in $\mathbb{Q}/\mathbb{Z}$. If $h(\beta) \le h(\gamma)$ then exactly one of the following holds:*

$$I(\gamma', \gamma'') \subseteq I(\beta', \beta) \quad \textit{or} \quad I(\gamma', \gamma'') \subseteq I(\beta, \beta'') \quad \textit{or} \quad I(\beta', \beta'') \cap I(\gamma', \gamma'') = \emptyset.$$

If $g_1(x)$ and $g_2(x)$ are functions in $L^2(\mathbb{R}/\mathbb{Z})$ we write

$$\langle g_1, g_2 \rangle = \int_{\mathbb{R}/\mathbb{Z}} g_1(x)\overline{g_2(x)}\,\mathrm{d}x \quad \text{and} \quad \|g_1\|_2 = \Big\{ \int_{\mathbb{R}/\mathbb{Z}} |g_1(x)|^2\,\mathrm{d}x \Big\}^{1/2}$$

for their inner product and norm, respectively.

For each point β in $\mathbb{Q}/\mathbb{Z}$ we define a normalized function $f_\beta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ of bounded variation as follows. If $\beta = 0$ we set $f_\beta(x) = 1$, and if $\beta \neq 0$ we set

$$f_\beta(x) = h(\beta)\psi(x-\beta) - h(\beta')\psi(x-\beta') - h(\beta'')\psi(x-\beta''). \tag{13}$$

As the integral of ψ over $\mathbb{R}/\mathbb{Z}$ is 0, it follows immediately that

$$\int_{\mathbb{R}/\mathbb{Z}} f_\beta(x)\,\mathrm{d}x = \begin{cases} 1 & \text{if } \beta = 0, \\ 0 & \text{if } \beta \neq 0. \end{cases} \tag{14}$$

If $h(\beta) = q \ge 2$, $h(\beta') = q'$, and $h(\beta'') = q''$, then a useful alternative definition of f_β is given by

$$f_\beta(x) = \begin{cases} q' & \text{if } x \in I(\beta', \beta), \\ -q'' & \text{if } x \in I(\beta, \beta''), \\ \frac{1}{2}q' & \text{if } x = \beta', \\ -\frac{1}{2}q'' & \text{if } x = \beta'', \\ \frac{1}{2}(q' - q'') & \text{if } x = \beta, \\ 0 & \text{if } x \notin \overline{I}(\beta', \beta''). \end{cases} \tag{15}$$

It is obvious from (15) that for $\beta \neq 0$ the function f_β is supported on the closed set $\overline{I}(\beta', \beta'')$. Also, using (15) we find that

$$\|f_\beta\|_2^2 = \int_{\mathbb{R}/\mathbb{Z}} f_\beta(x)^2\,\mathrm{d}x = \frac{(q')^2}{q'q} + \frac{(q'')^2}{qq''} = 1. \tag{16}$$

Thus each function f_β has norm 1 and $\langle f_0, f_\beta \rangle = 0$ for $\beta \neq 0$.

Now suppose that β and γ are distinct nonzero points of $\mathbb{Q}/\mathbb{Z}$. Without loss of generality we may assume that $h(\beta) \le h(\gamma)$. In view of Lemma 1 there are three cases to consider. If $I(\gamma', \gamma'') \subseteq I(\beta', \beta)$ then

$$\langle f_\beta, f_\gamma \rangle = \int_{(\gamma', \gamma'')} f_\beta(x) f_\gamma(x)\,\mathrm{d}x = f_\beta(\gamma) \int_{(\gamma', \gamma'')} f_\gamma(x)\,\mathrm{d}x = 0.$$

The other cases lead to the same conclusion in a similar manner. This shows that the collection of functions $\{f_\beta : \beta \in \mathbb{Q}/\mathbb{Z}\}$ forms an orthonormal subset of $L^2(\mathbb{R}/\mathbb{Z})$.

It remains now to show that the functions $\{f_\beta : \beta \in \mathbb{Q}/\mathbb{Z}\}$ form a complete orthonormal basis for $L^2(\mathbb{R}/\mathbb{Z})$. Toward this end let Q be a positive integer and define $K_Q : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ by

$$K_Q(x,y) = \sum_{\beta \in \mathcal{F}_Q} f_\beta(x) f_\beta(y). \tag{17}$$

Note that for each x the function $y \mapsto K_Q(x,y)$ is normalized and that for each y the function $x \mapsto K_Q(x,y)$ is normalized. For each $Q \geq 2$ we define a function $J_Q : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ by

$$J_Q(x,y) = \sum_{h(\beta)=Q} f_\beta(x) f_\beta(y).$$

For $Q \geq 2$ it is clear that

$$K_Q(x,y) = K_{Q-1}(x,y) + J_Q(x,y). \tag{18}$$

We also define a map

$$\sigma : (\mathbb{R}/\mathbb{Z} \setminus \mathbb{Q}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z} \setminus \mathbb{Q}/\mathbb{Z}) \to \{1,2,\ldots\} \cup \{\infty\}$$

as follows: if $x = y$ then $\sigma(x,y) = \infty$, and if x and y are distinct irrational points in $\mathbb{R}/\mathbb{Z}$ we define $\sigma(x,y)$ to be the smallest positive integer Q such that x and y are not in the same component interval of $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q$.

Lemma 2. *Let x and y be points in $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q$. If x and y belong to the same component interval $I(\gamma_1, \gamma_2)$ of $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q$ then we have*

$$K_Q(x,y) = h(\gamma_1) h(\gamma_2).$$

If x and y belong to distinct component intervals of $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q$ then

$$K_Q(x,y) = 0.$$

Proof. If x and y are irrational and $\sigma(x,y) \leq Q-1$ then it is easily seen that $J_Q(x,y) = 0$. If $\sigma(x,y) = Q$ then there exists a unique element β in $\mathcal{F}_Q \setminus \mathcal{F}_{Q-1}$ such that (after renaming x and y if necessary)

$$x \in I(\beta', \beta) \quad \text{and} \quad y \in I(\beta, \beta'').$$

It follows that $h(\beta) = Q$ and that $J_Q(x,y) = -h(\beta')h(\beta'')$. If $Q+1 \leq \sigma(x,y)$ then there exists a pair of consecutive points $\{\gamma_1, \gamma_2\}$ in $\mathcal{F}_Q$ such that both x and y belong to the component $I(\gamma_1, \gamma_2)$. In this case we find that

$$J_Q(x,y) = \begin{cases} h(\gamma_2)^2 & \text{if } h(\gamma_1) = Q, \\ h(\gamma_1)^2 & \text{if } h(\gamma_2) = Q, \\ 0 & \text{otherwise.} \end{cases}$$

Next we use this information about $J_Q(x,y)$ to determine $K_Q(x,y)$.

We argue by induction on Q. The case $Q = 1$ is trivial, so we assume that $Q \geq 2$ and that the assertion of the lemma holds for $K_{Q-1}(x, y)$. Now when x and y are irrational there are three cases to consider.

If $\sigma(x, y) \leq Q - 1$ then $J_Q(x, y) = 0$ and $K_{Q-1}(x, y) = 0$ by the inductive hypothesis. Hence $K_Q(x, y) = 0$ by (18).

If $\sigma(x, y) = Q$ then there exists a unique point β in $\mathcal{F}_Q \setminus \mathcal{F}_{Q-1}$ such that (after renaming x and y if necessary)

$$x \in I(\beta', \beta) \quad \text{and} \quad y \in I(\beta, \beta'').$$

We conclude that

$$J_Q(x, y) = -h(\beta')h(\beta''),$$

and by the inductive hypothesis

$$K_{Q-1}(x, y) = h(\beta')h(\beta'').$$

Again we find that $K_Q(x, y) = 0$ by (18).

Finally, if $Q + 1 \leq \sigma(x, y)$ then there exists a pair of consecutive points $\{\gamma_1, \gamma_2\}$ in $\mathcal{F}_Q$ such that both x and y belong to the component $I(\gamma_1, \gamma_2)$. If $h(\gamma_1) \leq Q - 1$ and $h(\gamma_2) \leq Q - 1$ then $J_Q(x, y) = 0$ and

$$K_{Q-1}(x, y) = h(\gamma_1)h(\gamma_2)$$

by the inductive hypothesis. If $h(\gamma_1) = Q$ then $J_Q(x, y) = h(\gamma_2)^2$. It follows that $\gamma_2 = \gamma_1''$ and therefore $\{\gamma_1', \gamma_1''\}$ are consecutive points in $\mathcal{F}_{Q-1}$. Thus we find that

$$K_{Q-1}(x, y) = h(\gamma_1')h(\gamma_1'')$$

by the inductive hypothesis, and we conclude that

$$K_Q(x, y) = h(\gamma_1')h(\gamma_1'') + h(\gamma_1'')^2 = h(\gamma_1)h(\gamma_1'') = h(\gamma_1)h(\gamma_2).$$

If $h(\gamma_2) = Q$ the argument is essentially the same. This proves the lemma when x and y are irrational. However, comparing (15) and (17) it is easy to see that the functions $x \mapsto K_Q(x, y)$ and $y \mapsto K_Q(x, y)$ are constant on the interior of all component intervals of $\mathcal{F}_Q$, so the result of the lemma extends immediately to all points x and y in $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q$. □

Lemma 3. *Let $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be an integrable function. Then for almost all points x in $\mathbb{R}/\mathbb{Z}$ we have*

$$\lim_{Q \to \infty} \int_{\mathbb{R}/\mathbb{Z}} g(y) K_Q(x, y)\ \mathrm{d}y = g(x).$$

Proof. By the Lebesgue density theorem we have

$$\lim_{z \to x} (z - x)^{-1} \int_x^z |g(y) - g(x)|\ \mathrm{d}y = 0 \tag{19}$$

for almost all x in $\mathbb{R}$. Assume that x is an irrational real number such that (19) holds. For each Q let $\{\beta_Q, \gamma_Q\}$ be consecutive points in $\mathcal{F}_Q$ such that

x belongs to $I(\beta_Q, \gamma_Q)$. Let β_Q and γ_Q be coset representatives such that $\beta_Q < x < \gamma_Q$ and $\gamma_Q - \beta_Q \leq Q^{-1}$. Then by Lemma 2 we have

$$\begin{aligned}
\left| \int_0^1 g(y) K_Q(x,y)\ \mathrm{d}y - g(x) \right| &= \left| (\gamma_Q - \beta_Q)^{-1} \int_{\beta_Q}^{\gamma_Q} (g(y) - g(x))\ \mathrm{d}y \right| \\
&\leq (\gamma_Q - \beta_Q)^{-1} \int_{\beta_Q}^{x} |g(y) - g(x)|\ \mathrm{d}y \\
&\quad + (\gamma_Q - \beta_Q)^{-1} \int_{x}^{\gamma_Q} |g(y) - g(x)|\ \mathrm{d}y \\
&\leq (x - \beta_Q)^{-1} \int_{\beta_Q}^{x} |g(y) - g(x)|\ \mathrm{d}y \\
&\quad + (\gamma_Q - x)^{-1} \int_{x}^{\gamma_Q} |g(y) - g(x)|\ \mathrm{d}y.
\end{aligned}$$

Because

$$\lim_{Q \to \infty} \beta_Q = x \quad \text{and} \quad \lim_{Q \to \infty} \gamma_Q = x,$$

the result follows from (19). □

Theorem 4. *The collection of functions $\{f_\beta : \beta \in \mathbb{Q}/\mathbb{Z}\}$ forms a complete, orthonormal basis for $L^2(\mathbb{R}/\mathbb{Z})$.*

Proof. Suppose that $g(x)$ is in $L^2(\mathbb{R}/\mathbb{Z})$ and $g(x)$ is orthogonal to each function $f_\beta(x)$. That is, we suppose that

$$\langle g, f_\beta \rangle = \int_{\mathbb{R}/\mathbb{Z}} g(y) f_\beta(y)\ \mathrm{d}y = 0$$

for each β in $\mathbb{Q}/\mathbb{Z}$. Then we have

$$\int_{\mathbb{R}/\mathbb{Z}} g(y) K_Q(x,y)\ \mathrm{d}y = \sum_{\beta \in \mathcal{F}_Q} \langle g, f_\beta \rangle f_\beta(x) = 0 \tag{20}$$

for all points x in $\mathbb{R}/\mathbb{Z}$. Letting $Q \to \infty$ in (20) and applying Lemma 3, it follows that $g(x)$ is 0 in $L^2(\mathbb{R}/\mathbb{Z})$. This shows that the collection $\{f_\beta\}$ is complete in $L^2(\mathbb{R}/\mathbb{Z})$. □

3. The Continued Fraction Interpretation

We return to consideration of the principal convergents and intermediate convergents from the continued fraction expansion of an irrational real number α. It will be convenient to assume that $0 < \alpha < 1$ and to write $\mathfrak{F}_Q$ for the set of Farey fractions in $[0, 1]$ of order Q. As is well known, the convergents and

intermediate convergents to α satisfy the following inequalities. If n is an odd positive integer we have

$$\frac{p_{n-1}}{q_{n-1}} < \alpha < \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} < \cdots < \frac{2p_{n-1} + p_{n-2}}{2q_{n-1} + q_{n-2}} < \frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} \tag{21}$$

and if n is an even positive integer then

$$\frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} < \frac{2p_{n-1} + p_{n-2}}{2q_{n-1} + q_{n-2}} < \cdots < \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} < \alpha < \frac{p_{n-1}}{q_{n-1}}. \tag{22}$$

Next we observe that for each positive integer Q there exists a unique pair of positive integers M and N such that

$$1 \le M \le a_N \quad \text{and} \quad Mq_{N-1} + q_{N-2} \le Q < (M+1)q_{N-1} + q_{N-2}. \tag{23}$$

If N is odd we have

$$\frac{p_{N-1}}{q_{N-1}} < \alpha < \frac{Mp_{N-1} + p_{N-2}}{Mq_{N-1} + q_{N-2}} \tag{24}$$

and if N is even then

$$\frac{Mp_{N-1} + p_{N-2}}{Mq_{N-1} + q_{N-2}} < \alpha < \frac{p_{N-1}}{q_{N-1}}. \tag{25}$$

Equations (24) and (25) determine the unique open Farey interval in $[0,1]\backslash\mathfrak{F}_Q$ that contains α.

Lemma 4. *Let α be an irrational point in $\mathbb{R}/\mathbb{Z}$ and let*

$$\beta = \frac{mp_{n-1} + p_{n-2}}{mq_{n-1} + q_{n-2}}, \quad \textit{where} \quad 1 \le m \le a_n,$$

be a nonzero point in $E_n(\alpha)$ for some positive integer n. If n is odd then

$$\beta' = \frac{p_{n-1}}{q_{n-1}}, \quad \beta'' = \frac{(m-1)p_{n-1} + p_{n-2}}{(m-1)q_{n-1} + q_{n-2}}, \quad \textit{and} \quad \alpha \in I(\beta', \beta). \tag{26}$$

If n is even then

$$\beta' = \frac{(m-1)p_{n-1} + p_{n-2}}{(m-1)q_{n-1} + q_{n-2}}, \quad \beta'' = \frac{p_{n-1}}{q_{n-1}}, \quad \textit{and} \quad \alpha \in I(\beta, \beta''). \tag{27}$$

Proof. If n is an odd positive integer then it follows from properties of Farey fractions that the three fractions

$$\frac{p_{n-1}}{q_{n-1}} < \frac{mp_{n-1} + p_{n-2}}{mq_{n-1} + q_{n-2}} < \frac{(m-1)p_{n-1} + p_{n-2}}{(m-1)q_{n-1} + q_{n-2}} \tag{28}$$

are consecutive points of $\mathfrak{F}_Q$ for $Q = mq_{n-1} + q_{n-2}$. (Note that $(m,n) \neq (1,1)$ because $\beta \neq 0$.) This verifies the identities for β' and β'' in (26). Then it follows from (21) that α belongs to $I(\beta', \beta)$. Similarly, if n is an even positive integer then the three fractions

$$\frac{(m-1)p_{n-1} + p_{n-2}}{(m-1)q_{n-1} + q_{n-2}} < \frac{mp_{n-1} + p_{n-2}}{mq_{n-1} + q_{n-2}} < \frac{p_{n-1}}{q_{n-1}} \tag{29}$$

are consecutive points of $\mathfrak{F}_Q$ for $Q = mq_{n-1} + q_{n-2}$. The assertions in (27) follow as in the previous case. $\square$

The functions $\{f_\beta\}$ were initially defined by (13) and (15). We now show that the value of $f_\beta(\alpha)$ depends in a simple way on the convergents and intermediate convergents to α.

Proof of Theorem 1. If $\beta = 0$ then β belongs to $E_1(\alpha)$ and (4) is obvious. Assume that $\beta \neq 0$ belongs to E_n and that

$$\beta = \frac{mp_{n-1} + p_{n-2}}{mq_{n-1} + q_{n-2}} \quad \text{where} \quad 1 \leq m \leq a_n.$$

If n is odd then it follows from (26) that α belongs to the component interval $I(\beta', \beta)$. From the definition of f_β we conclude that

$$f_\beta(\alpha) = q_{n-1} = (-1)^{n-1} q_{n-1}.$$

Similarly, if n is even then (27) implies that α belongs to the component interval $I(\beta, \beta'')$. In this case we find that

$$f_\beta(\alpha) = -q_{n-1} = (-1)^{n-1} q_{n-1}.$$

Now assume that $f_\beta(\alpha) \neq 0$. If $\beta = 0$ then β belongs to E_1. Otherwise we have either

$$\alpha \in I(\beta', \beta) \quad \text{or} \quad \alpha \in I(\beta, \beta''). \tag{30}$$

Write $h(\beta) = Q$ and as in (23) let M and N be the unique positive integers such that

$$1 \leq M \leq a_N \quad \text{and} \quad Mq_{N-1} + q_{N-2} \leq Q < (M+1)q_{N-1} + q_{N-2}.$$

If N is odd then (24) and (30) imply that

$$\beta = \frac{Mp_{N-1} + p_{N-2}}{Mq_{N-1} + q_{N-2}}, \tag{31}$$

and this shows that β belongs to E_N. Similarly, if N is even then (25) and (30) imply that (31) holds, and again we conclude that β belongs to E_N. $\square$

Certain partial sums involving the functions f_β also have a natural Diophantine interpretation. For each positive integer Q we define two functions

$$L_Q : \mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q \to \{1, 2, \ldots, Q\} \quad \text{and} \quad R_Q : \mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q \to \{1, 2, \ldots, Q\}.$$

If α is a point in $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q$ then there exists a unique pair $\{\gamma_1, \gamma_2\}$ of consecutive points in $\mathcal{F}_Q$ such that α belongs to $I(\gamma_1, \gamma_2)$. We define $L_Q(\alpha) = h(\gamma_1)$ and $R_Q(\alpha) = h(\gamma_2)$. From (17) and Lemma 2 we obtain the identity

$$1 + \sum_{2 \leq h(\beta) \leq Q} f_\beta(\alpha)^2 = L_Q(\alpha) R_Q(\alpha) \tag{32}$$

for α in $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q$. We now establish some further identities of this sort.

Lemma 5. *If* $\alpha \in \mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q$ *then*

$$2 + \sum_{2 \le h(\beta) \le Q} |f_\beta(\alpha)| = R_Q(\alpha) + L_Q(\alpha), \tag{33}$$

and

$$\sum_{2 \le h(\beta) \le Q} f_\beta(\alpha) = R_Q(\alpha) - L_Q(\alpha). \tag{34}$$

Proof. As in the proof of Lemma 2, it suffices to establish (33) and (34) for α irrational. For $n = 1, 2, \ldots$ let E_n be the collection of convergents and intermediate convergents defined by (2). Let M and N be the unique positive integers such that (23) holds. If N is odd then from (24) we find that

$$\{\gamma_1, \gamma_2\} = \left\{ \frac{p_{N-1}}{q_{N-1}}, \frac{Mp_{N-1} + p_{N-2}}{Mq_{N-1} + q_{N-2}} \right\}$$

are consecutive points in $\mathcal{F}_Q$ such that α belongs to $I(\gamma_1, \gamma_2)$. This implies that

$$L_Q(\alpha) = q_{N-1} \quad \text{and} \quad R_Q(\alpha) = Mq_{N-1} + q_{N-2}. \tag{35}$$

If N is even then (25) implies that

$$\{\gamma_1, \gamma_2\} = \left\{ \frac{Mp_{N-1} + p_{N-2}}{Mq_{N-1} + q_{N-2}}, \frac{p_{N-1}}{q_{N-1}} \right\}$$

are consecutive points in $\mathcal{F}_Q$ such that α belongs to $I(\gamma_1, \gamma_2)$. In this case we conclude that

$$L_Q(\alpha) = Mq_{N-1} + q_{N-2} \quad \text{and} \quad R_Q(\alpha) = q_{N-1}. \tag{36}$$

For each $m \in \{1, 2, \ldots, a_N\}$ write

$$\beta_m = \frac{mp_{N-1} + p_{N-2}}{mq_{N-1} + q_{N-2}}.$$

Using (4), (35), and (36) we find that

$$\begin{aligned}
\sum_{h(\beta) \le Q} |f_\beta(\alpha)| &= \sum_{n=1}^{N-1} \sum_{\beta \in E_n} |f_\beta(\alpha)| + \sum_{m=1}^{M} |f_{\beta_m}(\alpha)| \\
&= \sum_{n=1}^{N-1} \sum_{\beta \in E_n} q_{n-1} + \sum_{m=1}^{M} q_{N-1} \\
&= \sum_{n=1}^{N-1} a_n q_{n-1} + Mq_{N-1} \qquad (37) \\
&= \sum_{n=1}^{N-1} (q_n - q_{n-2}) + Mq_{N-1}
\end{aligned}$$

$$
\begin{aligned}
&= q_{N-1} + Mq_{N-1} + q_{N-2} - 1 \\
&= R_Q(\alpha) + L_Q(\alpha) - 1.
\end{aligned}
$$

As $f_0(\alpha) = 1$, it is clear that (37) is equivalent to (33).

In a similar manner, using (35) and (36) we get

$$
\begin{aligned}
\sum_{h(\beta)\le Q} f_\beta(\alpha) &= \sum_{n=1}^{N-1} \sum_{\beta\in E_n} f_\beta(\alpha) + \sum_{m=1}^{M} f_{\beta_m}(\alpha) \\
&= \sum_{n=1}^{N-1} \sum_{\beta\in E_n} (-1)^{n-1} q_{n-1} + \sum_{m=1}^{M} (-1)^{N-1} q_{N-1} \\
&= \sum_{n=1}^{N-1} (-1)^{n-1} a_n q_{n-1} + (-1)^{N-1} M q_{N-1} \\
&= \sum_{n=1}^{N-1} (-1)^{n-1} (q_n - q_{n-2}) + (-1)^{N-1} M q_{N-1} \\
&= (-1)^{N-1}\{Mq_{N-1} + q_{N-2} - q_{N-1}\} + 1 \\
&= R_Q(\alpha) - L_Q(\alpha) + 1,
\end{aligned}
$$

which proves (34). □

Corollary 1. *If α belongs to $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q$ then*

$$
1 + \sum_{2\le h(\beta)\le Q} f_\beta^+(\alpha) = R_Q(\alpha) \quad \text{and} \quad 1 + \sum_{2\le h(\beta)\le Q} f_\beta^-(\alpha) = L_Q(\alpha).
$$

Proof. Identity (33) can be written as

$$
\Big\{1 + \sum_{2\le h(\beta)\le Q} f_\beta^+(\alpha)\Big\} + \Big\{1 + \sum_{2\le h(\beta)\le Q} f_\beta^-(\alpha)\Big\} = R_Q(\alpha) + L_Q(\alpha), \tag{38}
$$

and identity (34) can be written as

$$
\Big\{1 + \sum_{2\le h(\beta)\le Q} f_\beta^+(\alpha)\Big\} - \Big\{1 + \sum_{2\le h(\beta)\le Q} f_\beta^-(\alpha)\Big\} = R_Q(\alpha) - L_Q(\alpha). \tag{39}
$$

The statement of the corollary now plainly follows from (38) and (39). □

The argument used to prove Theorem 1 can be applied to other functions indexed by points β in $\mathbb{Q}/\mathbb{Z}$ and supported on $\overline{I}(\beta', \beta'')$. As an example we define a further collection of real valued functions $\{\chi_\beta : \beta \in \mathbb{Q}/\mathbb{Z}\}$ with domain $\mathbb{R}/\mathbb{Z}$ as follows. For $\beta = 0$ we set $\chi_0(x) = 1$ for all x in $\mathbb{R}/\mathbb{Z}$. Then for $\beta \ne 0$ we set

$$
\chi_\beta(x) = \begin{cases} 1 & \text{if } x \in I(\beta', \beta''), \\ \frac{1}{2} & \text{if } x = \beta' \text{ or } x = \beta'', \\ 0 & \text{if } x \notin \overline{I}(\beta', \beta''). \end{cases}
$$

For each nonzero point β in $\mathbb{Q}/\mathbb{Z}$ the function $\chi_\beta(x)$ is the normalized characteristic function of the component interval $I(\beta', \beta'')$. If β_1 and β_2 are distinct points in $\mathbb{Q}/\mathbb{Z}$ with $h(\beta_1) = h(\beta_2) \geq 2$, then the open component intervals $I(\beta_1', \beta_1'')$ and $I(\beta_2', \beta_2'')$ are disjoint. Thus for each positive integer q the normalized characteristic function of the subset

$$\bigcup_{h(\beta)=q} I(\beta', \beta'').$$

is the function $X_q : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ defined by

$$X_q(x) = \sum_{h(\beta)=q} \chi_\beta(x). \tag{40}$$

Theorem 5. *Let α be an irrational point in $\mathbb{R}/\mathbb{Z}$, and for $n = 1, 2, \ldots$ let E_n be the collection of convergents and intermediate convergents defined by (2). If β is in $\mathbb{Q}/\mathbb{Z}$ then $\chi_\beta(\alpha) = 1$ if and only if β belongs to E_n for some $n = 1, 2, \ldots$. Moreover, the sum*

$$\sum_{h(\beta)\leq Q} \chi_\beta(\alpha) = \sum_{q=1}^{Q} X_q(\alpha)$$

is exactly the number of convergents and intermediate convergents to α with height less than or equal to Q.

Proof. The first assertion of the corollary follows as in the proof of Theorem 1. For the second assertion let M and N be the unique positive integers such that

$$1 \leq M \leq a_N \quad \text{and} \quad Mq_{N-1} + q_{N-2} \leq Q < (M+1)q_{N-1} + q_{N-2},$$

and for each $m \in \{1, 2, \ldots, a_N\}$ write

$$\beta_m = \frac{mp_{N-1} + p_{N-2}}{mq_{N-1} + q_{N-2}}.$$

Then we have

$$\begin{aligned} \sum_{h(\beta)\leq Q} \chi_\beta(\alpha) &= \sum_{n=1}^{N-1} \sum_{\beta \in E_n} \chi_\beta(\alpha) + \sum_{m=1}^{M} \chi_{\beta_m}(\alpha) \\ &= \sum_{n=1}^{N-1} \sum_{\beta \in E_n} 1 + M \\ &= \sum_{n=1}^{N-1} a_n + M. \end{aligned}$$

Plainly, this is the number of convergents and intermediate convergents to α with height less than or equal to Q. $\square$

Let $\mathcal{Q}$ be a subset of positive integers. For each irrational point α in $\mathbb{R}/\mathbb{Z}$ write

$$\mathcal{D}(\alpha) = \{mq_{n-1} + q_{n-2} : 1 \le m \le a_n \text{ and } 1 \le n\}$$

for the set of denominators from the collection of convergents and intermediate convergents to α. Arguing as in the proof of Theorem 5, we find that

$$\sum_{q \in \mathcal{Q}} X_q(\alpha) = \left|\mathcal{D}(\alpha) \cap \mathcal{Q}\right|. \tag{41}$$

If the set $\mathcal{Q}$ is such that the integral

$$\int_{\mathbb{R}/\mathbb{Z}} \Big\{ \sum_{q \in \mathcal{Q}} X_q(x) \Big\} \,\mathrm{d}x \tag{42}$$

is finite then the integrand is finite for almost all x, and therefore (41) is finite for almost all irrational points α in $\mathbb{R}/\mathbb{Z}$. We conjecture that if the integral (42) is infinite then (41) is infinite for almost all irrational points α. The situation is clarified by the following simple estimate.

Lemma 6. *For each integer $q \ge 2$ we have*

$$\int_{\mathbb{R}/\mathbb{Z}} X_q(x) \,\mathrm{d}x = \frac{2\varphi(q)}{q^2} \Big\{ \log q + \sum_{p|q} \frac{\log p}{p-1} + c_0 \Big\} + O\Big(\frac{\log\log q}{q^2}\Big), \tag{43}$$

where c_0 is Euler's constant, and the sum on the right of (43) *is over prime numbers p that divide q.*

Proof. Suppose that $\beta = a/q$, where $1 \le a < q$ and $(a,q) = 1$. Write $\overline{a}$ for the unique integer such that $1 \le \overline{a} < q$ and $a\overline{a} \equiv 1 \pmod{q}$. We find that $h(\beta') = \overline{a}$ and $h(\beta'') = q - \overline{a}$, and therefore

$$\begin{aligned} \int_{\mathbb{R}/\mathbb{Z}} X_q(x) \,\mathrm{d}x &= \sum_{\substack{\beta \in \mathbb{Q}/\mathbb{Z} \\ h(\beta) = q}} \frac{1}{h(\beta')h(\beta'')} = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{1}{\overline{a}(q-\overline{a})} \\ &= \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Big(\frac{1}{\overline{a}q} + \frac{1}{(q-\overline{a})q} \Big) \\ &= \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{2}{aq}. \end{aligned} \tag{44}$$

Then using Möbius inversion and well known estimates we have

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{1}{a} = \sum_{a=1}^{q} \frac{1}{a} \sum_{\substack{d|q \\ d|a}} \mu(d) = \sum_{d|q} \mu(d) \sum_{\substack{a=1 \\ d|a}}^{q} \frac{1}{a} = \sum_{d|q} \frac{\mu(d)}{d} \sum_{b=1}^{q/d} \frac{1}{b}$$

$$= \sum_{d|q} \frac{\mu(d)}{d} \Big\{ \log q - \log d + c_0 + \frac{d}{2q} + O\Big(\frac{d^2}{q^2}\Big) \Big\} \tag{45}$$

$$= \frac{\varphi(q) \log q}{q} - \sum_{d|q} \frac{\mu(d) \log d}{d} + \frac{c_0 \varphi(q)}{q} + O\Big(\frac{\log \log q}{q}\Big).$$

The statement of the lemma follows now by combining (44), (45), and the basic identity

$$-\sum_{d|q} \frac{\mu(d) \log d}{d} = \frac{\varphi(q)}{q} \sum_{p|q} \frac{\log p}{p-1}. \tag{46}$$

□

It follows from the estimate (43) that the integral (42) is infinite if and only if the series

$$\sum_{q \in \mathcal{Q}} \frac{\varphi(q) \log q}{q^2} \tag{47}$$

diverges. Thus we state our conjecture in the following form.

Conjecture 1. *Let $\mathcal{Q}$ be a subset of positive integers. Then the set $\mathcal{D}(\alpha) \cap \mathcal{Q}$ is infinite for almost all irrational points α in $\mathbb{R}/\mathbb{Z}$ if and only if the series* (47) *diverges.*

We note that the analogous statement for the principal convergents to almost all α is a well known theorem of Erdös [3]. Related questions of Diophantine approximation are considered in [8, Chapter 2] and [13].

4. Sequences of Martingale Differences

Let $n \mapsto \beta_n$ be a bijective map from the set $\mathbb{N}$ of positive integers onto the group $\mathbb{Q}/\mathbb{Z}$. Then we say that $\beta_1, \beta_2, \dots$ is an *enumeration* of the elements of $\mathbb{Q}/\mathbb{Z}$. For each positive integer n we define $\mathcal{B}_n$ to be the finite σ-algebra of subsets of $\mathbb{R}/\mathbb{Z}$ generated by the components of the open set

$$\mathbb{R}/\mathbb{Z} \setminus \{\beta_1, \beta_2, \dots, \beta_n\} \tag{48}$$

together with the collection of singleton sets $\{\beta_1\}, \{\beta_2\}, \dots, \{\beta_n\}$. Thus a function $g : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is $\mathcal{B}_n$-measurable if and only if it is constant on each component of the open set (48). Clearly we have $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \cdots$.

In this section we determine a simple arithmetic condition that classifies all orderings $\beta_1, \beta_2, \dots$ such that f_{β_n} is $\mathcal{B}_n$-measurable for each $n = 1, 2, \dots$ and the sequence of functions and σ-algebras

$$\{(f_{\beta_n}, \mathcal{B}_n) : n = 1, 2, \dots\} \tag{49}$$

forms a sequence of martingale differences. The martingale differences which arise from this construction are a special case of a general class of such functions considered by R. F. Gundy [6]. These observations allow us to exploit results from the theory of martingales to obtain metric theorems about the continued fraction expansion of almost all real numbers.

Again let $\beta_1, \beta_2, \ldots$ be an enumeration of the elements of $\mathbb{Q}/\mathbb{Z}$. We say that this enumeration is *admissible* if it satisfies the following three conditions:

(i) $\beta_1 = 0$,
(ii) if $m \geq 2$ and $\beta_k = \beta'_m$ then $k < m$,
(iii) if $m \geq 2$ and $\beta_l = \beta''_m$ then $l < m$.

It follows easily that an admissible enumeration of $\mathbb{Q}/\mathbb{Z}$ must begin with either $0, \frac{1}{2}, \frac{1}{3}, \ldots$, or with $0, \frac{1}{2}, \frac{2}{3}, \ldots$.

Suppose that $\beta_1, \beta_2, \ldots$ is an enumeration of $\mathbb{Q}/\mathbb{Z}$ such that $n \mapsto h(\beta_n)$ is nondecreasing. Then $\beta_1 = 0$, and using (11) we find that if β_n is nonzero then $h(\beta'_n) < h(\beta_n)$ and $h(\beta''_n) < h(\beta_n)$. It follows that such an enumeration is admissible. As an example, if we enumerate $\mathbb{Q}/\mathbb{Z}$ as

$$\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \ldots, \tag{50}$$

then $n \mapsto h(\beta_n)$ is nondecreasing and the enumeration is admissible. Thus admissible enumerations of $\mathbb{Q}/\mathbb{Z}$ certainly exist.

The admissible enumeration (50) is constructed by arranging the points of $\mathbb{Q}/\mathbb{Z}$ in order of increasing height, and then ordering points of equal height by ordering their coset representatives in $(0, 1)$. This construction also leads to an admissible ordering for other naturally occurring functions on $\mathbb{Q}/\mathbb{Z}$. We describe such an admissible enumeration associated to the Stern-Brocot tree (see [5]). If β is a rational number but not an integer then β has exactly two finite continued fraction expansions. One expansion has the form

$$\beta = [a_0; a_1, a_2, \ldots, a_{N-1}, a_N], \quad \text{where } a_N \geq 2, \tag{51}$$

and then the other expansion is

$$\beta = [a_0; a_1, a_2, \ldots, a_{N-1}, a_N - 1, 1].$$

We define

$$s(\beta) = a_1 + a_2 + \cdots + a_N,$$

so that $s(\beta)$ is the sum of the partial quotients (other than a_0) in both expansions of β. If n is an integer we define $s(n) = 1$. It is clear that s is constant on cosets of $\mathbb{Q}/\mathbb{Z}$ and thus it is well defined as a map $s : \mathbb{Q}/\mathbb{Z} \to \mathbb{N}$. Clearly $s(\beta) = 1$ if and only if $\beta = 0$ in $\mathbb{Q}/\mathbb{Z}$.

Suppose that β is rational number but not an integer, and β has the finite continued fraction expansion (51). If N is odd, then arguing as in the proof of Lemma 4, we get

$$s(\beta') + a_N = s(\beta) \quad \text{and} \quad s(\beta'') + 1 = s(\beta).$$

If N is even we find that

$$s(\beta') + 1 = s(\beta) \quad \text{and} \quad s(\beta'') + a_N = s(\beta).$$

In particular, these identities show that if β is a nonzero point in $\mathbb{Q}/\mathbb{Z}$, then $s(\beta') < s(\beta)$ and $s(\beta'') < s(\beta)$. It follows that if $\beta_1, \beta_2, \dots$ is an enumeration of $\mathbb{Q}/\mathbb{Z}$ such that $n \mapsto s(\beta_n)$ is nondecreasing, then the enumeration is admissible. For example, the enumeration

$$\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{7}, \tfrac{3}{8}, \tfrac{3}{7}, \tfrac{4}{7}, \tfrac{5}{8}, \tfrac{5}{7}, \tfrac{4}{5}, \dots, \tag{52}$$

that corresponds to the ordering induced by the Stern-Brocot tree, is such that $n \mapsto s(\beta_n)$ is nondecreasing. Hence (52) is an admissible enumeration of $\mathbb{Q}/\mathbb{Z}$, but distinct from (50). In particular, the map $n \mapsto h(\beta_n)$ fails to be nondecreasing for the enumeration (52).

Theorem 6. *Let $\beta_1, \beta_2, \dots$ be an enumeration of $\mathbb{Q}/\mathbb{Z}$. Then $\beta_1, \beta_2, \dots$ is admissible if and only if for each positive integer n the function f_{β_n} is $\mathcal{B}_n$-measurable.*

Proof. First assume that $\beta_1, \beta_2, \dots$ is admissible. As $\beta_1 = 0$ it follows that $f_{\beta_1} = f_0$ is constant. Hence it is trivial that f_{β_1} is $\mathcal{B}_1$-measurable. Now suppose that $n \geq 2$. By hypothesis both β_n' and β_n'' are elements of the set $\{\beta_1, \beta_2, \dots, \beta_n\}$. Therefore $\mathcal{B}_n$ contains the sub-σ-algebra $\mathcal{A}_n$ generated by the components of the open set

$$\mathbb{R}/\mathbb{Z} \setminus \{\beta_n, \beta_n', \beta_n''\}$$

and the singleton sets $\{\beta_n\}, \{\beta_n'\}, \{\beta_n''\}$. From the definition (15) it follows that f_{β_n} is $\mathcal{A}_n$-measurable and hence also $\mathcal{B}_n$-measurable.

Now assume that the function f_{β_n} is $\mathcal{B}_n$-measurable for each positive integer n. In particular the function f_{β_1} must be constant on each component of the open set $\mathbb{R}/\mathbb{Z} \setminus \{\beta_1\}$. That is, f_{β_1} must be constant on $\mathbb{R}/\mathbb{Z} \setminus \{\beta_1\}$. Hence f_{β_1} is constant almost everywhere and therefore $\beta_1 = 0$. Now assume that $n \geq 2$. Then f_{β_n} is constant on each of the open sets $I(\beta_n', \beta_n)$ and $I(\beta_n, \beta_n'')$. Therefore the σ-algebra $\mathcal{A}_n$ defined above is the smallest σ-algebra for which f_{β_n} is measurable. It follows that $\mathcal{A}_n \subseteq \mathcal{B}_n$ and therefore β_n' and β_n'' must be elements of the set $\{\beta_1, \beta_2, \dots, \beta_n\}$. That is, if $\beta_k = \beta_n'$ then $k < n$, and if $\beta_l = \beta_n''$ then $l < n$. This shows that the enumeration $\beta_1, \beta_2, \dots$ is admissible. $\square$

Lemma 7. *Let $\beta_1, \beta_2, \dots$ be an admissible enumeration of $\mathbb{Q}/\mathbb{Z}$. If β_l and β_m are distinct nonzero points in $\mathbb{Q}/\mathbb{Z}$ such that β_m is contained in $I(\beta_l', \beta_l'')$, then $l < m$.*

Proof. Let $h(\beta_l) = Q \geq 2$ and define

$$M(\beta_l) = \{n \in \mathbb{N} : \beta_n \in I(\beta_l', \beta_l'')\}.$$

Then $M(\beta_l)$ is not empty and we may clearly assume that m is the smallest positive integer in $M(\beta_l)$. Now $I(\beta_l', \beta_l'')$ is a component interval in $\mathbb{R}/\mathbb{Z}\backslash\mathcal{F}_{Q-1}$ and therefore $h(\beta_l) = Q \leq h(\beta_m)$. As

$$\beta_m \in I(\beta_l', \beta_l'') \cap I(\beta_m', \beta_m''),$$

it follows from Lemma 1 that either

$$I(\beta_m', \beta_m'') \subseteq I(\beta_l', \beta_l) \quad \text{or} \quad I(\beta_m', \beta_m'') \subseteq I(\beta_l, \beta_l'').$$

Assume that the first alternative

$$I(\beta_m', \beta_m'') \subseteq I(\beta_l', \beta_l) \tag{53}$$

holds. Write $\beta_m' = \beta_j$ and $\beta_m'' = \beta_k$. As the enumeration $\beta_1, \beta_2, \ldots$ is admissible we have $j < m$ and $k < m$. Therefore neither β_m' nor β_m'' can belong to $I(\beta_l', \beta_l)$. From (53) we conclude that

$$\beta_m' = \beta_l' \quad \text{and} \quad \beta_m'' = \beta_l. \tag{54}$$

Since $\beta_1, \beta_2, \ldots$ is admissible, the second identity in (54) implies that $l < m$. If the second alternative

$$I(\beta_m', \beta_m'') \subseteq I(\beta_l, \beta_l'')$$

holds then the inequality $l < m$ follows in a similar manner. This proves the lemma. □

Next we recall that (49) is a sequence of martingale differences if each function f_{β_n} is $\mathcal{B}_n$-measurable, and if for $n = 1, 2, \ldots$ the conditional expectation of $f_{\beta_{n+1}}$ with respect to $\mathcal{B}_n$ is 0 almost everywhere. In the present setting, the conditional expectation of $f_{\beta_{n+1}}$ with respect to $\mathcal{B}_n$ is 0 almost everywhere if and only if

$$\int_J f_{\beta_{n+1}}(x)\, \mathrm{d}x = 0 \tag{55}$$

for each component J of the open set $\mathbb{R}/\mathbb{Z} \setminus \{\beta_1, \beta_2, \ldots, \beta_n\}$.

Theorem 7. *Let $\beta_1, \beta_2, \ldots$ be an admissible enumeration of $\mathbb{Q}/\mathbb{Z}$. Then the sequence of functions and σ-algebras*

$$\{(f_{\beta_n}, \mathcal{B}_n) : n = 1, 2, \ldots\} \tag{56}$$

is a sequence of martingale differences.

Proof. Let n be a positive integer. Then $\beta_{n+1} \neq 0$ and by hypothesis the points β_{n+1}' and β_{n+1}'' are contained in the set $\{\beta_1, \beta_2, \ldots, \beta_n\}$. Obviously $\beta_1 = 0$ is not contained in the open set $I(\beta_{n+1}', \beta_{n+1}'')$. If $2 \leq m \leq n$ then by Lemma 7 the point β_m is not contained in $I(\beta_{n+1}', \beta_{n+1}'')$. It follows that $I(\beta_{n+1}', \beta_{n+1}'')$ is a component of the open set

$$\mathbb{R}/\mathbb{Z} \setminus \{\beta_1, \beta_2, \ldots, \beta_n\}. \tag{57}$$

Using the definition (15) we find that

$$\int_{I(\beta'_{n+1},\beta''_{n+1})} f_{\beta_{n+1}}(x)\ \mathrm{d}x = 0.$$

As $f_{\beta_{n+1}}$ is supported on $\overline{I}(\beta'_{n+1},\beta''_{n+1})$, it follows that (55) also holds whenever $J \neq I(\beta'_{n+1},\beta''_{n+1})$ is any other component of (57). This proves the theorem. $\square$

R. F. Gundy [6] investigated a general class of martingale differences called H-systems. If $\beta_1,\beta_2,\dots$ is an admissible enumeration of $\mathbb{Q}/\mathbb{Z}$, then it follows from Theorem 4 and [6, Proposition 1.1] that (56) is an example of an H-system. The following theorem is [6, Theorem 2.1(a)] applied to the sequence (56).

Theorem 8. *Let $F:\mathbb{R}/\mathbb{Z}\to\mathbb{R}\cup\{\pm\infty\}$ be a Borel measurable function that is finite almost everywhere and let $\beta_1,\beta_2,\dots$ be an admissible enumeration of $\mathbb{Q}/\mathbb{Z}$. Then there exist real numbers $\{c(\beta_n): n=1,2,\dots\}$ such that*

$$\lim_{N\to\infty}\sum_{n=1}^{N} c(\beta_n) f_{\beta_n}(x) = F(x)$$

for almost all x in $\mathbb{R}/\mathbb{Z}$.

In order to express results of this sort in terms of the continued fraction expansion of α, it is convenient to select the admissible ordering of $\mathbb{Q}/\mathbb{Z}$ so that it has some arithmetical significance. Here we use the admissible ordering (50).

Proof of Theorem 2. Let $\beta_1,\beta_2,\dots$ be the admissible enumeration (50). By Theorem 8 there exist real numbers $\{c(\beta):\beta\in\mathbb{Q}/\mathbb{Z}\}$ such that

$$\lim_{Q\to\infty}\sum_{h(\beta)\le Q} c(\beta) f_\beta(x) = F(x) \tag{58}$$

for almost all x in $\mathbb{R}/\mathbb{Z}$. Assume that α is an irrational point in $\mathbb{R}/\mathbb{Z}$ such that (58) holds. For each positive integer Q let $M=M(Q,\alpha)$ and $N=N(Q,\alpha)$ be the unique positive integers defined by (23). Using Theorem 1 we can write

$$\begin{aligned}\sum_{h(\beta)\le Q} c(\beta) f_\beta(\alpha) &= \sum_{n=1}^{N-1}(-1)^{n-1} q_{n-1}(\alpha)\sum_{\beta\in E_n(\alpha)} c(\beta)\\ &\quad + (-1)^{N-1} q_{N-1}(\alpha)\sum_{m=1}^{M} c\Big(\frac{mp_{N-1}(\alpha)+p_{N-2}(\alpha)}{mq_{N-1}(\alpha)+q_{N-2}(\alpha)}\Big).\end{aligned} \tag{59}$$

We restrict the parameter Q in (59) to the subsequence of denominators of the convergents to α. Along this subsequence the positive integers M and N are related by the identity $M=a_N$. Thus (59) reduces to the simpler assertion (5). $\square$

Next we construct an example to show that the numbers $\{c(\beta) : \beta \in \mathbb{Q}/\mathbb{Z}\}$ that occur in the statement of Theorem 2 are not uniquely determined by F. We require the following lemmas, the first of which identifies the inverse of the maps $\beta \mapsto \beta'$ and $\beta \mapsto \beta''$.

Lemma 8. *Let $r'/s' < r/s < r''/s''$ be three consecutive points in the set $\mathfrak{F}_s$ of Farey fractions of order s. Then we have*

$$\{\beta \in \mathbb{Q}/\mathbb{Z} : \beta' = r/s\} = \Big\{\frac{mr + r''}{ms + s''} : m = 1, 2, \dots\Big\} \tag{60}$$

and

$$\{\gamma \in \mathbb{Q}/\mathbb{Z} : \gamma'' = r/s\} = \Big\{\frac{nr + r'}{ns + s'} : n = 1, 2, \dots\Big\}. \tag{61}$$

Proof. The identity (60) follows immediately from the inequalities

$$\frac{r}{s} < \cdots < \frac{mr + r''}{ms + s''} < \frac{(m-1)r + r''}{(m-1)s + s''} < \cdots < \frac{r + r''}{s + s''} < \frac{r''}{s''},$$

and basic properties of Farey fractions. Then (61) is proved in the same manner. □

Lemma 9. *Let $r'/s' < r/s < r''/s''$ be three consecutive points in the set $\mathfrak{F}_s$ and write $\delta = r/s$ for the image of r/s in $\mathbb{Q}/\mathbb{Z}$. Define*

$$\beta_m = \frac{mr + r''}{ms + s''} \quad \textit{for} \quad m = 0, 1, 2, \dots,$$

and

$$\gamma_n = \frac{nr + r'}{ns + s'} \quad \textit{for} \quad n = 0, 1, 2, \dots.$$

Then $\delta' = r'/s'$ and $\delta'' = r''/s''$ in $\mathbb{Q}/\mathbb{Z}$, and for positive integers M and N we have

$$\sum_{m=1}^{M} f_{\beta_m}(x) = \begin{cases} Mh(\delta) & \textit{if } x \in I(\delta, \beta_M), \\ -h(\delta'') & \textit{if } x \in I(\beta_M, \delta''), \\ 0 & \textit{if } x \notin \overline{I}(\delta, \delta''). \end{cases} \tag{62}$$

and

$$\sum_{n=1}^{N} f_{\gamma_n}(x) = \begin{cases} -Nh(\delta) & \textit{if } x \in I(\gamma_N, \delta), \\ h(\delta') & \textit{if } x \in I(\delta', \gamma_N), \\ 0 & \textit{if } x \notin \overline{I}(\delta', \delta). \end{cases} \tag{63}$$

Proof. That $\delta' = r'/s'$ and $\delta'' = r''/s''$ in $\mathbb{Q}/\mathbb{Z}$ follows the definition of the maps $\beta \mapsto \beta'$ and $\beta \mapsto \beta''$, and the hypothesis that $r'/s' < r/s < r''/s''$ are consecutive points in $\mathfrak{F}_s$. From the definition (13) we get

$$\sum_{m=1}^{M} f_{\beta_m}(x) = \sum_{m=1}^{M} h(\beta_m)\psi(x - \beta_m) \tag{64}$$

$$- \sum_{m=1}^{M} h(\beta'_m)\psi(x-\beta'_m) - \sum_{m=1}^{M} h(\beta''_m)\psi(x-\beta''_m).$$

It follows using Lemma 8 that $\beta'_m = \delta$, $\beta''_m = \beta_{m-1}$ for each $m = 1, 2, \ldots, M$, and $\beta_0 = \delta''$. These observations allow us to simplify (64). We find that

$$\sum_{m=1}^{M} f_{\beta_m}(x) = h(\beta_M)\psi(x-\beta_M) - Mh(\delta)\psi(x-\delta) - h(\delta'')\psi(x-\delta''),$$

and this easily implies (62). The identity (63) is established in a similar manner. □

Assume now that δ is a nonzero point in $\mathbb{Q}/\mathbb{Z}$. Lemma 8 and Lemma 9 imply that

$$\lim_{Q\to\infty} \Big\{ f_\delta(x) - \sum_{\substack{h(\beta)\le Q\\ \delta=\beta'}} f_\beta(x) - \sum_{\substack{h(\gamma)\le Q\\ \delta=\gamma''}} f_\gamma(x) \Big\} = 0 \tag{65}$$

at almost all points x in $\mathbb{R}/\mathbb{Z}$. This shows that the numbers $\{c(\beta) : \beta \in \mathbb{Q}/\mathbb{Z}\}$ in the statement of Theorem 2 are not uniquely determined by the function F.

5. Further Applications of the Martingale Property

In this section we formalize the argument used to prove Theorem 2 and derive further results about the continued fraction expansion of almost all irrational points α in $\mathbb{R}/\mathbb{Z}$.

Let $\beta_1, \beta_2, \ldots$ be an admissible enumeration of $\mathbb{Q}/\mathbb{Z}$ and let $\{c(\beta_n) : n = 1, 2, \ldots\}$ be a collection of real numbers. For $N = 1, 2, \ldots$ we define $S_N : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ by

$$S_N(x) = \sum_{n=1}^{N} c(\beta_n) f_{\beta_n}(x). \tag{66}$$

Theorem 6 implies that each function S_N is $\mathcal{B}_N$-measurable, and from Theorem 7 we conclude that the sequence of functions and σ-algebras

$$\big\{(S_N, \mathcal{B}_N) : N = 1, 2, \ldots\big\} \tag{67}$$

forms a martingale. Of course this fact allows us to draw conclusions about the behavior of the partial sums $S_N(x)$ for almost all x as $N \to \infty$. As before we wish to express our results in terms of the continued fraction expansion of α. Therefore we use the admissible ordering (50) and consider the subsequence of partial sums

$$T_Q(x) = \sum_{h(\beta)\le Q} c(\beta) f_\beta(x). \tag{68}$$

Clearly we have

$$T_Q(x) = S_N(x), \quad \text{where} \quad N = N_Q = \sum_{q \le Q} \varphi(q).$$

It will be convenient to write $\mathcal{M}_Q = \mathcal{B}_{N_Q}$ for the corresponding σ-algebra and $\mathcal{M}_0 = \{\emptyset, \mathbb{R}/\mathbb{Z}\}$ for the trivial σ-algebra. Thus a function $g : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is $\mathcal{M}_Q$-measurable if it is measurable with respect to the σ-algebra generated by the component intervals of $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q$ together with the singleton sets $\{\beta\}$ for β in $\mathcal{F}_Q$. Alternatively, g is $\mathcal{M}_Q$-measurable if it is constant on each component interval of $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_Q$. It follows that the sequence of functions and σ-algebras

$$\big\{(T_Q, \mathcal{M}_Q) : Q = 1, 2, \ldots\big\} \tag{69}$$

forms a martingale. It is instructive to note that if $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is an integrable function, then the function (using a standard notation)

$$g(x|\mathcal{M}_Q) = \int_{\mathbb{R}/\mathbb{Z}} K_Q(x, y) g(y) \, \mathrm{d}y$$

is the conditional expectation of g given the σ-algebra $\mathcal{M}_Q$. This is easily verified using Lemma 2.

The following result is an application of Theorem 1 and the martingale convergence theorem.

Theorem 9. *Let $\{c(\beta) : \beta \in \mathbb{Q}/\mathbb{Z}\}$ be a collection of real numbers and for each positive integer Q let $T_Q(x)$ be defined by* (68)*. If the sequence of L^1-norms*

$$\int_{\mathbb{R}/\mathbb{Z}} \big|T_Q(x)\big| \, \mathrm{d}x, \quad Q = 1, 2, \ldots, \tag{70}$$

is bounded, then

$$\lim_{N \to \infty} \sum_{n=1}^{N} (-1)^{n-1} q_{n-1}(\alpha) \sum_{\beta \in E_n(\alpha)} c(\beta) = F(\alpha) \tag{71}$$

exists for almost all irrational points α in $\mathbb{R}/\mathbb{Z}$ and $\|F\|_1 < \infty$.

Proof. By the martingale convergence theorem (see [2, Section 4.2] or [4, Section 3.2]), the hypothesis (70) implies that the limit

$$\lim_{Q \to \infty} T_Q(x) = F(x) \tag{72}$$

exists for almost all x in $\mathbb{R}/\mathbb{Z}$ and satisfies $\|F\|_1 < \infty$. Suppose that α is an irrational point in $\mathbb{R}/\mathbb{Z}$ for which (72) holds. For each positive integer Q let $M = M(Q, \alpha)$ and $N = N(Q, \alpha)$ be the unique positive integers defined by (23). Using Theorem 1 we can write

$$T_Q(\alpha) = \sum_{n=1}^{N-1} (-1)^{n-1} q_{n-1}(\alpha) \sum_{\beta \in E_n(\alpha)} c(\beta)$$

$$+ (-1)^{N-1} q_{N-1}(\alpha) \sum_{m=1}^{M} c\Big(\frac{m p_{N-1}(\alpha) + p_{N-2}(\alpha)}{m q_{N-1}(\alpha) + q_{N-2}(\alpha)}\Big). \tag{73}$$

If $Q = q_N(\alpha)$ is a denominator of a convergent to α then (73) simplifies to

$$T_{q_N(\alpha)}(\alpha) = \sum_{n=1}^{N} (-1)^{n-1} q_{n-1}(\alpha) \sum_{\beta \in E_n(\alpha)} c(\beta). \tag{74}$$

Now (71) follows from (72) and (74). □

If $F : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is an integrable function and T_Q is given by

$$T_Q(x) = \int_{\mathbb{R}/\mathbb{Z}} K_Q(x, y) F(y)\ \mathrm{d}y = \sum_{h(\beta) \le Q} c(\beta) f_\beta(x), \tag{75}$$

where

$$c(\beta) = \int_{\mathbb{R}/\mathbb{Z}} f_\beta(y) F(y)\ \mathrm{d}y,$$

then the conclusion (71) follows from the much simpler Lemma 3. It is well known (see [2, Theorem 5.6, Section 4.5]) that T_Q has the form (75) for some integrable function F if and only if T_Q converges to F in L^1-norm as $Q \to \infty$. By appealing to the martingale convergence theorem we are able to establish (71) under the weaker hypothesis (70).

Let δ be a nonzero point in $\mathbb{Q}/\mathbb{Z}$. Using (62) and (63) we find that

$$\int_{\mathbb{R}/\mathbb{Z}} \Big| f_\delta(x) - \sum_{\substack{h(\beta) \le Q \\ \delta = \beta'}} f_\beta(x) - \sum_{\substack{h(\gamma) \le Q \\ \delta = \gamma''}} f_\gamma(x) \Big|\ \mathrm{d}x = 2h(\delta)^{-1},$$

and

$$\int_{\mathbb{R}/\mathbb{Z}} \Big| \sum_{\substack{h(\beta) \le Q \\ \delta = \beta'}} f_\beta(x) + \sum_{\substack{h(\gamma) \le Q \\ \delta = \gamma''}} f_\gamma(x) \Big|\ \mathrm{d}x \le 4h(\delta)^{-1},$$

for all positive integers Q. Thus

$$T_Q(x) = \sum_{\substack{h(\beta) \le Q \\ \delta = \beta'}} f_\beta(x) + \sum_{\substack{h(\gamma) \le Q \\ \delta = \gamma''}} f_\gamma(x)$$

is an example that does not have the form (75), but for which the sequence (70) of L^1-norms is bounded. Of course in this example we can establish the almost everywhere convergence (65) directly without appealing to Theorem 9.

For the remainder of this section we consider the behavior of the partial sums $T_Q(x)$ under the hypothesis that $\beta \mapsto c(\beta) h(\beta)$ is bounded on $\mathbb{Q}/\mathbb{Z}$. It

will be convenient to write

$$T_Q(x) = \sum_{q=1}^{Q} U_q(x), \quad \text{where} \quad U_q(x) = \sum_{h(\beta)=q} c(\beta) f_\beta(x).$$

If β_1 and β_2 are distinct points in $\mathbb{Q}/\mathbb{Z}$ with $h(\beta_1) = h(\beta_2)$ then $\overline{I}(\beta_1', \beta_1'')$ and $\overline{I}(\beta_2', \beta_2'')$ intersect in a finite set, and therefore in a set of measure zero. From this observation and the definition (15) we find that

$$\begin{aligned} \tfrac{1}{2} q \max\{|c(\beta)| : h(\beta) = q\} &\leq \sup_{x \in \mathbb{R}/\mathbb{Z}} \bigl|U_q(x)\bigr| \\ &= \|U_q\|_\infty \leq q \max\{|c(\beta)| : h(\beta) = q\}. \end{aligned} \tag{76}$$

This shows that $\beta \mapsto c(\beta)h(\beta)$ is bounded on $\mathbb{Q}/\mathbb{Z}$ if and only if the sequence $\|U_q\|_\infty$ is bounded for $q = 1, 2, \ldots$.

Lemma 10. *For $q = 1, 2, \ldots, Q$, let $A_q \subseteq \mathbb{R}/\mathbb{Z}$ be an $\mathcal{M}_{q-1}$-measurable subset, where $\mathcal{M}_0 = \{\emptyset, \mathbb{R}/\mathbb{Z}\}$ is the trivial σ-algebra. Write χ_{A_q} for the characteristic function of A_q. Then we have*

$$\int_{\mathbb{R}/\mathbb{Z}} \Bigl\{ \sum_{q=1}^{Q} \chi_{A_q}(x) U_q(x) \Bigr\}^2 \, \mathrm{d}x = \int_{\mathbb{R}/\mathbb{Z}} \Bigl\{ \sum_{q=1}^{Q} \chi_{A_q}(x) U_q(x)^2 \Bigr\} \, \mathrm{d}x. \tag{77}$$

Proof. We square out the integrand on the left of (77) and integrate term by term. Then it is clear that the lemma will follow if we can verify that

$$\int_{A_q \cap A_r} U_q(x) U_r(x) \, \mathrm{d}x = 0 \tag{78}$$

whenever $1 \leq q < r \leq Q$. Let J be a component of $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_{r-1}$. Then we have

$$\int_J U_r(x) \, \mathrm{d}x = 0.$$

The function U_q is $\mathcal{M}_{r-1}$-measurable and therefore constant on J. This implies that

$$\int_J U_q(x) U_r(x) \, \mathrm{d}x = 0. \tag{79}$$

By hypothesis the set $A_q \cap A_r$ is $\mathcal{M}_{r-1}$-measurable. Therefore it can be written as a finite disjoint union of component intervals of $\mathbb{R}/\mathbb{Z} \setminus \mathcal{F}_{r-1}$ together with a set of measure zero. Thus (78) follows immediately from (79), and the lemma is proved. $\square$

Theorem 10. *Let $\{c(\beta) : \beta \in \mathbb{Q}/\mathbb{Z}\}$ be a collection of real numbers such that $\beta \mapsto c(\beta)h(\beta)$ is bounded on $\mathbb{Q}/\mathbb{Z}$. Write $\mathcal{C}$ for the subset of points α in $\mathbb{R}/\mathbb{Z}$*

such that

$$\lim_{Q\to\infty}\sum_{q=1}^{Q} U_q(\alpha) \tag{80}$$

exists and is finite. Write $\mathcal{D}$ for the subset of points α in $\mathbb{R}/\mathbb{Z}$ such that both

$$\liminf_{Q\to\infty}\sum_{q=1}^{Q} U_q(\alpha) = -\infty \quad \textit{and} \quad \limsup_{Q\to\infty}\sum_{q=1}^{Q} U_q(\alpha) = +\infty. \tag{81}$$

Write $\mathcal{E}$ for the subset of points α in $\mathbb{R}/\mathbb{Z}$ such that

$$\sum_{q=1}^{\infty} U_q(\alpha)^2 < \infty. \tag{82}$$

Then we have

$$\text{(i) } |\mathcal{C}\cup\mathcal{D}| = 1, \quad \text{(ii) } |\mathcal{C}\setminus\mathcal{E}| = 0, \quad \textit{and} \quad \text{(iii) } |\mathcal{E}\setminus\mathcal{C}| = 0. \tag{83}$$

Proof. The sequence (69) forms a martingale, and by hypothesis the increments

$$\sup_{x\in\mathbb{R}/\mathbb{Z}}\bigl|T_q(x) - T_{q-1}(x)\bigr| = \|U_q\|_\infty$$

are bounded for $q = 1, 2, \ldots$. Therefore (i) follows from [2, Theorem 3.1, Section 4.3].

For positive integers L and Q let

$$A(L,Q) = \{x\in\mathbb{R}/\mathbb{Z} : |T_q(x)| < L \text{ for } q = 1, 2, \ldots, Q-1\},$$

so that

$$A(L,1) = \mathbb{R}/\mathbb{Z} \supseteq A(L,2) \supseteq A(L,3) \supseteq \cdots .$$

It follows that $A(L,Q)$ is $\mathcal{M}_{Q-1}$-measurable and

$$A(L,\infty) = \bigcap_{Q=1}^{\infty} A(L,Q) = \{x\in\mathbb{R}/\mathbb{Z} : |T_Q(x)| < L \text{ for all } Q\geq 1\}.$$

Clearly we have

$$\mathcal{C} \subseteq \bigcup_{L=1}^{\infty} A(L,\infty). \tag{84}$$

Next we define the stopping time $\eta_L : \mathbb{R}/\mathbb{Z} \to \mathbb{N}\cup\infty$ by

$$\eta_L(x) = \min\{Q : |T_Q(x)| \geq L\},$$

where $\eta_L(x) = \infty$ if x belongs to $A(L,\infty)$. Then we define

$$T_Q^{(\eta_L(x))}(x) = \sum_{q=1}^{\min\{\eta_L(x),Q\}} U_q(x) = \sum_{q=1}^{Q} \chi_{A(L,q)}(x) U_q(x),$$

so that for each positive integer L the sequence

$$\left\{(T_Q^{(\eta_L)}, \mathcal{M}_Q) : Q = 1, 2, \dots\right\}$$

forms a martingale, (see [1, 17.6 Corollary 2]). And by Lemma 10 we have

$$\int_{\mathbb{R}/\mathbb{Z}} \left\{T_Q^{(\eta_L(x))}(x)\right\}^2 \,\mathrm{d}x = \int_{\mathbb{R}/\mathbb{Z}} \Big\{\sum_{q=1}^{Q} \chi_{A(L,q)}(x) U_q(x)^2\Big\} \,\mathrm{d}x. \tag{85}$$

The inequality (76) and the assumption that $\beta \mapsto c(\beta)h(\beta)$ is bounded on $\mathbb{Q}/\mathbb{Z}$ imply that

$$\sup\{\|U_q\|_\infty : q = 1, 2, \dots\} = M < \infty. \tag{86}$$

Thus we have

$$\left|T_Q^{(\eta_L(x))}(x)\right| \le L + M \tag{87}$$

uniformly for all x in $\mathbb{R}/\mathbb{Z}$ and all $Q = 1, 2, \dots$. From the martingale convergence theorem we conclude that

$$\lim_{Q\to\infty} T_Q^{(\eta_L(x))}(x) = V_L(x)$$

exists for almost all points x in $\mathbb{R}/\mathbb{Z}$. From (87) and the dominated convergence theorem we obtain

$$\lim_{Q\to\infty} \int_{\mathbb{R}/\mathbb{Z}} \left\{T_Q^{(\eta_L(x))}(x)\right\}^2 \,\mathrm{d}x = \int_{\mathbb{R}/\mathbb{Z}} V_L(x)^2 \,\mathrm{d}x < \infty.$$

Then from (85) we find that

$$\lim_{Q\to\infty} \int_{\mathbb{R}/\mathbb{Z}} \left\{T_Q^{(\eta_L(x))}(x)\right\}^2 \,\mathrm{d}x = \int_{\mathbb{R}/\mathbb{Z}} \Big\{\sum_{q=1}^{\infty} \chi_{A(L,q)}(x) U_q(x)^2\Big\} \,\mathrm{d}x < \infty. \tag{88}$$

Now (88) implies that

$$\sum_{q=1}^{\infty} \chi_{A(L,q)}(x) U_q(x)^2 < \infty$$

for almost all x in $\mathbb{R}/\mathbb{Z}$. Hence we have

$$\sum_{q=1}^{\infty} U_q(x)^2 < \infty \tag{89}$$

for almost all x in $A(L,\infty)$. Then (84) implies that (89) holds for almost all points x in $\mathcal{C}$. This proves (ii).

The proof of (iii) is very similar. For positive integers L and Q let

$$B(L,Q) = \{x \in \mathbb{R}/\mathbb{Z} : \sum_{q=1}^{Q-1} U_q(x)^2 < L\},$$

so that

$$B(L,1) = \mathbb{R}/\mathbb{Z} \supseteq B(L,2) \supseteq B(L,3) \supseteq \cdots .$$

It follows that $B(L,Q)$ is $\mathcal{M}_{Q-1}$-measurable and

$$B(L,\infty) = \bigcap_{Q=1}^{\infty} B(L,Q) = \{x \in \mathbb{R}/\mathbb{Z} : \sum_{q=1}^{Q-1} U_q(x)^2 < L \text{ for all } Q \geq 1\}.$$

And we have

$$\mathcal{E} \subseteq \bigcup_{L=1}^{\infty} B(L,\infty). \tag{90}$$

In this case we define a stopping time $\tau_L : \mathbb{R}/\mathbb{Z} \to \mathbb{N} \cup \infty$ by

$$\tau_L(x) = \min\{Q : \sum_{q=1}^{Q} U_q(x)^2 \geq L\},$$

where $\tau_L(x) = \infty$ if x belongs to $B(L,\infty)$. Then we write

$$T_Q^{(\tau_L(x))}(x) = \sum_{q=1}^{\min\{\tau_L(x),Q\}} U_q(x) = \sum_{q=1}^{Q} \chi_{B(L,q)}(x) U_q(x),$$

so that for each positive integer L the sequence

$$\big\{(T_Q^{(\tau_L)}, \mathcal{M}_Q) : Q = 1,2,\dots\big\}$$

forms a martingale, (see [1, 17.6 Corollary 2]). By Lemma 10 we have

$$\int_{\mathbb{R}/\mathbb{Z}} \big\{T_Q^{(\tau_L(x))}(x)\big\}^2 \, \mathrm{d}x = \int_{\mathbb{R}/\mathbb{Z}} \Big\{ \sum_{q=1}^{Q} \chi_{B(L,q)}(x) U_q(x)^2 \Big\} \, \mathrm{d}x. \tag{91}$$

The bound (86) implies that

$$\sum_{q=1}^{Q} \chi_{B(L,q)}(x) U_q(x)^2 \leq L + M^2$$

uniformly for all x in $\mathbb{R}/\mathbb{Z}$ and $Q = 1,2,\dots$. It follows from (91) and the martingale convergence theorem that

$$\lim_{Q\to\infty} T_Q^{(\tau_L(x))}(x) = W_L(x)$$

exists and is finite for almost all points x in $\mathbb{R}/\mathbb{Z}$. Hence the limit

$$\lim_{Q\to\infty} T_Q(x) \tag{92}$$

exists and is finite for almost all points x in $B(L,\infty)$. Now (90) implies that the limit (92) exists and is finite for almost all points x in $\mathcal{E}$. This proves (iii). □

The conclusions (ii) and (iii) in the statement of Theorem 10 are essentially the same as those obtained by Gundy [6, Theorem 3.1] under somewhat different hypotheses. In particular, Gundy works with an H-system and a *regular* sequence of σ-algebras. The sequence of σ-algebras $\mathcal{M}_Q$, $Q = 1, 2, \ldots$, is not regular in Gundy's sense, but we are able to establish the same type of result by using instead the hypothesis that $\beta \mapsto c(\beta)h(\beta)$ is bounded on $\mathbb{Q}/\mathbb{Z}$.

Finally, the series that occur in Theorem 10 can be rewritten using the continued fraction interpretation of the functions f_β. In this way we arrive at the statement of Theorem 3.

References

[1] H. Bauer, *Probability Theory*, Walter de Gruyter, New York, 1996.
[2] R. Durrett, *Probability: Theory and Examples*, Duxbury Press, third edition, 2005.
[3] P. Erdös, On the distribution of convergents of almost all real numbers, *J. Number Theory*, Vol. 2, (1970), 425–441.
[4] A. M. Garsia, *Topics in Almost Everywhere Convergence*, Lectures in Advanced Mathematics, 4, Markham, Chicago, 1970.
[5] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, New York, 1994.
[6] R. F. Gundy, Martingale theory and pointwise convergence of certain orthogonal series, *Trans. Amer. Math. Soc.*, Vol. 124, (1966), 228–248.
[7] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, fifth edition, 1979.
[8] G. Harman, *Metric Number Theory*, London Mathematical Society Monographs New Series Oxford University Press Inc., New York, 1998.
[9] M. Hata, Farey fractions and sums over coprime pairs, *Acta Arith.*, 70(2), (1995), 149–159.
[10] A. Ya. Khinchin, *Continued Fractions*, University of Chicago Press, Chicago, (1964).
[11] A. M. Rockett and P. Szüsz, *Continued Fractions*, World Scientific, Singapore, 1992.
[12] W. M. Schmidt, *Diophantine Approximation*, Lecture Notes in Mathematics 785, Springer-Verlag, New York, 1980.
[13] J. D. Vaaler, On the metric theory of Diophantine approximation, *Pacific J. Math.*, Vol. 76, 527–539.

Department of Mathematics, University of Texas, Austin, Texas 78712 USA
E-mail address: `ahaynes@math.utexas.edu`
E-mail address: `vaaler@math.utexas.edu`